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# Low-temperature properties of ferromagnetic Fibonacci superlattices

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**Abstract.** Theoretical analysis of the layered quasi-periodic Fibonacci structures (superlattices-sequence) is presented for the systems consisting of $n_A$ and $n_B$ ferromagnetically ordered planes within the layers with $S_a$ and $S_b$ spins, respectively, while the interfaces are coupled with bilinear and/or biquadratic exchange interaction, within the framework of localized spin model in the low-temperature limit. Transfer matrix method and direct diagonalization after the bosonization in Bloch's approximation resulted both in the same analytical expression for the magnon-excitation energy. The equivalence (at low-temperatures) of the transfer matrix (spin) and boson approach was discussed, as well as the role of the interlayer biquadratic coupling between different blocks constituting the Fibonacci sequences. Also, our approach allows the determination of the internal energy and calculation of the magnon contribution to the specific heat. It was clearly demonstrated that the magnon specific heat vanishes for $T \to 0$. Our results are compared with the results of other authors.

**PACS.** 71.70.Gm Exchange interactions – 75.30.Ds Spin waves – 75.30.Hx Magnetic impurity interactions

## 1 Introduction

There are situations in modern physics where the theoretical studies on various models interplay with technological developments allowing the realization of materials simulating these models and the experimental studies of these materials. Such case occurred with quasiperiodic structures, 1d structures arranged in a particular manner. In fact, there we encounter so called superlattices where the set of planes with particular physical properties is arranged in a particular manner along the preferred direction. This can be realized by modern techniques of producing layered materials [1–3]. There are several interesting systems that could be realized and it seems worthwhile to study their properties, their energy spectrum in particular.

We are going to concentrate here on the problem of magnetic Fibonacci lattice, more precisely on a quasiperiodic structure whose magnetic cells are constructed following Fibonacci series: $A$, $AB$, $ABA$, $ABAAB, ...$ i.e. on the mapping $A \to AB$, $B \to A$. This problem was the subject of numerous studies [4,6–10,5], yet it is our opinion that there appear certain serious inconsistencies in the previous work, so we wish to reconsider the problem here.

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The planes are ferromagnetically ordered and interact ferromagnetically. We first noticed that there exists some problem with accounting the contribution of the biquadratic interaction. This lead us to reexamine the transfer matrix (TM) method calculations and obtain rather different results. In order to test these results, we performed the calculation within boson picture and obtained the same results. This approach allows us the calculation of the specific heat using Bose-Einstein statistics.

The basic model is introduced in Section 2 together with TM calculations. Section 3 is dedicated to the justification of the use of Bose-Einstein statistics for the calculation of thermodynamic properties. Numerical calculations of the consequences of our results are derived in Section 4, with the comparison with the results of the previous studies performed. Conclusions are in the Section 5.

Let us briefly comment here on possible extensions of the model. The planes are ferromagnetically ordered and interact ferromagnetically, both interactions described by the Heisenberg Hamiltonian, with biquadratic interaction added at the AB interfaces. The character of interaction can be modified by inserting the non-magnetic planes, or adding the spin dipole exchange [11]. Another extension of the model is by introducing antiferromagnetic interaction [12].

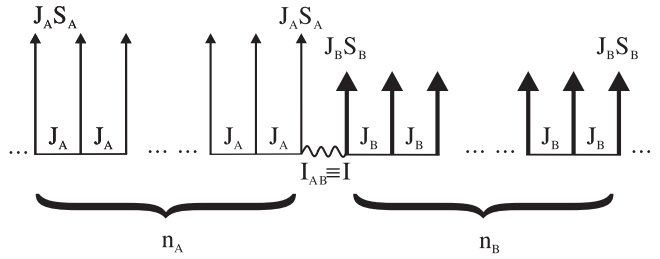


**Fig. 1.** Schematic representation of ferromagnetic superlattices (notation defined in text).

## 2 The model and transfer matrix calculations

Our aim is to study the series of superlattices, each one consisting of periodically repeating cells which represent one generation in Fibonacci series (Fibonacci generation = FG). We assume that the underlying lattice is a simple cubic one, with lattice constant $a$. We then consider a set of parallel planes with $z$-axis oriented orthogonal to the planes. The planes are built up of ferromagnetically (FM) interacting ions with spins $S_A$ (A-layers) and $S_B$ (B-layers). We consider only the nearest neighbors interaction and assume that the spins within the planes and between the planes of the same type interact by the common (bilinear) Heisenberg interaction, while at AB interface, the spin can interact by both bilinear and biquadratic interaction. A-spins interact by the exchange interaction $J_A$, B-spins by $J_B$. However, spins do form cells, and the interaction between two cells with boundary A-planes is $I_{AA} = I_A$ and with different boundary planes is $I_{AB} = I$. A closer look at Fibonacci series shows that there are no cell contacts with two B-planes, so this is the complete set of interactions. At AB interface, we shall also add later the biquadratic term.

Above mentioned superlattices are constructed in the following manner. A-spins are appearing in the blocks of $n_A$ planes, while B-spins occur in blocks of $n_B$ planes. We represent each term in Fibonacci series by a superlattice where A term is represented by a block of A-planes and B-term by the block of B-planes. The cells that are repeated, are consisting of a given term. The Figure 1 shows two different terms schemattically.

The total number of blocks within the $\nu$th FG is equal to the number of A-blocks and B-blocks:

$$g(\nu) = g_A(\nu) + g_B(\nu). \tag{1}$$

This number is defined by the basic relations:

$$\begin{aligned} g(\nu) &= g(\nu-1) + g(\nu-2); \;\; g_A(\nu) = g(\nu-1) \\ g_B(\nu) &= g(\nu-2) \end{aligned} \tag{2}$$

so that the total number of planes within one cell of $\nu$th FG equals:

$$N_\nu = g_A(\nu)\, n_A + g_B(\nu)\, n_B = g(\nu-1)\, n_A + g(\nu-2)\, n_B. \tag{3}$$

It is not common to write down the complete Hamiltonian, yet we shall do it in order to be able to perform the bosonization completely. The Hamiltonian describing superlattice corresponding to $\nu$th generation is given by:

$$\begin{aligned} \hat{H} = &-\frac{1}{2}\sum_m \sum_{n=1}^{N_\nu} J_n \sum_{\boldsymbol{\varrho}\,\vec{\delta}_\parallel} \left\{ \frac{1}{2}\left[\hat{S}^-_{m\boldsymbol{\varrho}}(n)\hat{S}^+_{m\boldsymbol{\varrho}+\boldsymbol{\delta}_\parallel}(n) + H.c.\right] \right. \\ &\left. + \hat{S}^z_{m\boldsymbol{\varrho}}(n)\hat{S}^z_{m\boldsymbol{\varrho}+\boldsymbol{\delta}_\parallel}(n) \right\} \\ &- \sum_m \sum_{\boldsymbol{\varrho}} \left\{ \sum_{n=1}^{N_\nu - 1} I_{n,n+1} \left[ \frac{1}{2}\left( \hat{S}^-_{m\boldsymbol{\varrho}}(n)\hat{S}^+_{m\boldsymbol{\varrho}}(n+1) + H.c. \right) \right. \right. \\ &\left. \left. + \hat{S}^z_{m\boldsymbol{\varrho}}(n)\hat{S}^z_{m\boldsymbol{\varrho}}(n+1) \right] \right. \\ &+ \frac{1}{2} I_{N_\nu,1} \left[ \frac{1}{2}\left( \hat{S}^-_{m\boldsymbol{\varrho}}(N_\nu)\hat{S}^+_{m+1\boldsymbol{\varrho}}(1) + \hat{S}^-_{m-1\boldsymbol{\varrho}}(N_\nu)\hat{S}^+_{m\boldsymbol{\varrho}}(1) + H.c. \right) \right. \\ &\left. \left. \left. + \hat{S}^z_{m\boldsymbol{\varrho}}(N_\nu)\hat{S}^z_{m+1\boldsymbol{\varrho}}(1) + \hat{S}^z_{m-1\boldsymbol{\varrho}}(N_\nu)\hat{S}^z_{m\boldsymbol{\varrho}}(1) \right] \right\}. \end{aligned} \tag{4}$$

The subscript "$m$" enumerates the cells, the subscript "$n$" enumerates the planes within the cell. The position of the spin within the plane is specified by $\boldsymbol{\rho}$ and $\boldsymbol{\delta}_{||}$ corresponds to the four nearest neighbors within the plane. As mentioned above, $J_n$ takes the values $J_A$ or $J_B$, $I_{n,n+1}$ takes the value $J_A$ within A-block, and $J_B$ within B-block, $I_A$ between two A-blocks and $I$ between A-block and B-block, irrespective of the order. We assume that $I_{N_\nu,N_\nu+1} = I_{N_\nu,1} = I_A$ or $I$. In order to simplify the notation, we consider $\hat{\boldsymbol{S}}(n)$ to be $\hat{\boldsymbol{S}}_A$ or $\hat{\boldsymbol{S}}_B$ depending on the block.

Now we sketch briefly the procedure established by Barnas [13] and generalized in our previous work [14]. One performs the Fourier-transformation within the plane, which is then characterized by the wave-vector $\boldsymbol{k}_{||}$. The equations of motion are evaluated for the spin operators $\hat{S}^+_{A/B}$ and linearized with $\hat{S}^z_{A/B} \to S_{A/B}$. Operators are then substituted by spin-wave amplitudes. Using this set of equations, the transfer matrices $\hat{M}_n$ can be introduced following the procedure elaborated in Section 2 of reference [14]. The procedure presented there is the generalization of the procedure proposed previously [6–9], since our approach allows even for the situation with one plane per block, while procedure of [6–9] can not be applied for blocks with less than three planes (so that at least one of the planes has to be the "bulk" one). The equivalence of the two approaches for $n \geq 3$ in $AB$ generation is easily proved.

We are interested in the manner in which the matrices $\hat{T}_\nu$ related to the superlattice cell of $\nu$th FG, can be constructed. It turns out that all such matrices can be divided into two groups.

The first group consists of the matrix

$$\hat{T}_{AA}(A) = \hat{M}_{AA}(r)\ \hat{M}_A^{n_A-2}\ \hat{M}_{AA}(l);$$
$$\hat{M}_A = \begin{pmatrix} \dfrac{\varepsilon_A - E}{S_A J_A} & -1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

where $\hat{M}_A$ is the "bulk" transfer matrix of A-block with $\varepsilon_A = 4J_A S_A(1-\gamma_{\boldsymbol{k}_\parallel}) + 2J_A S_A$, $\gamma_{\boldsymbol{k}_\parallel} = \frac{1}{2}(\cos k_x a + \cos k_y a)$ and

$$\hat{M}_{AA}(r) = \begin{pmatrix} \dfrac{\varepsilon_{AA} - E}{S_A I_A} & -\dfrac{J_A}{I_A} \\ 1 & 0 \end{pmatrix};$$
$$\hat{M}_{AA}(l) = \begin{pmatrix} \dfrac{\varepsilon_{AA} - E}{S_A J_A} & -\dfrac{I_A}{J_A} \\ 1 & 0 \end{pmatrix} \tag{6}$$

are right and left boundary matrices, respectively, with $\varepsilon_{AA} = 4J_A S_A(1-\gamma_{\boldsymbol{k}_\parallel}) + S_A(J_A + I_A)$. It is essential to stress that this matrix appears only in the first FG, where only A-blocks are present.

Transfer matrices of all other generations can be expressed in terms of four basic matrices. These are:

1) $\hat{T}_{AA}(B) = \hat{M}_{BA}(B)\ \hat{M}_B^{n_B-2}\ \hat{M}_{AB}(B)$ with the "bulk" matrix of B-block

$$\hat{M}_B = \begin{pmatrix} \dfrac{\varepsilon_B - E}{S_B J_B} & -1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

and $\varepsilon_B = 4J_B S_B(1-\gamma_{\boldsymbol{k}_\parallel}) + 2J_B S_B$ and two "boundary" matrices:

$$\hat{M}_{BA}(B) = \begin{pmatrix} \dfrac{\varepsilon_{BA}(B) - E}{S_B I} & -\dfrac{J_B}{I} \\ 1 & 0 \end{pmatrix};$$
$$\hat{M}_{AB}(B) = \begin{pmatrix} \dfrac{\varepsilon_{AB}(B) - E}{S_B J_B} & -\dfrac{I}{J_B} \\ 1 & 0 \end{pmatrix} \tag{8}$$

and $\varepsilon_{AB}(B) = \varepsilon_{BA}(B) = 4J_B S_B(1-\gamma_{\boldsymbol{k}_\parallel}) + J_B S_B + I S_A$. This matrix corresponds to B-block which is always situated between two A-blocks.

2) $\hat{T}_{BB}(A) = \hat{M}_{AB}(A)\ \hat{M}_A^{n_A-2}\ \hat{M}_{BA}(A)$ with boundary matrices

$$\hat{M}_{BA}(A) = \begin{pmatrix} \dfrac{\varepsilon_{BA}(A) - E}{S_A J_A} & -\dfrac{I}{J_A} \\ 1 & 0 \end{pmatrix};$$
$$\hat{M}_{AB}(A) = \begin{pmatrix} \dfrac{\varepsilon_{AB}(A) - E}{S_A I} & -\dfrac{J_A}{I} \\ 1 & 0 \end{pmatrix} \tag{9}$$

and $\varepsilon_{AB}(A) = \varepsilon_{BA}(A) = 4J_A S_A(1-\gamma_{\boldsymbol{k}_\parallel}) + J_A S_A + I S_B$. This matrix corresponds to A-block which is situated between two B-blocks.

3) $\hat{T}_{AB}(A) = \hat{M}_{AB}(A)\ \hat{M}_A^{n_A-2}\ \hat{M}_{AA}(l)$; 4) $\hat{T}_{BA}(A) = \hat{M}_{AA}(r)\ \hat{M}_A^{n_A-2}\ \hat{M}_{BA}(A)$.

These two matrices correspond to A-block situated between A-block and B-block in two different orders.

Using these four matrices, one can write the transfer matrices of several first Fibonacci generations:

$$\hat{T}_1 = \hat{T}_{AA}(A);\ \ \hat{T}_2 = \hat{T}_{AA}(B)\hat{T}_{BB}(A);$$
$$\hat{T}_3 = \hat{T}_{BA}(A)\hat{T}_{AA}(B)\hat{T}_{AB}(A).$$

Taking into account two important facts: $\hat{T}_{AA}(A) \neq \hat{T}_{BA}(A)$ and $\hat{T}_{BB}(A) \neq \hat{T}_{AB}(A)$ we obtain an important result $\hat{T}_3 \neq \hat{T}_1 \cdot \hat{T}_2$. If we continue, we obtain $\hat{T}_4 = \hat{T}_{AA}(B)\ \hat{T}_{AB}(A)\ \hat{T}_{BA}(A)\ \hat{T}_{AA}(B)\ \hat{T}_{BB}(A)$, $\hat{T}_5 = \hat{T}_{BA}(A)\ \hat{T}_{AA}(B)\ \hat{T}_{BB}(A)\ \hat{T}_{AA}(B)\ \hat{T}_{AB}(A)\ \hat{T}_{BA}(A)\ \hat{T}_{AA}(B)\ \hat{T}_{AB}(A)$. It can be easily concluded that $\hat{T}_4 \neq \hat{T}_2 \cdot \hat{T}_3$ and $\hat{T}_5 \neq \hat{T}_3 \cdot \hat{T}_4$, since boundary matrices are different.

This implies that practically none of the results presented in previous works [6–9] based on the recurrent relation

$$\hat{T}_\nu = \hat{T}_{\nu-2} \cdot \hat{T}_{\nu-1} \tag{10}$$

is valid. Further consequences of this fact will be analyzed later.

Following [14], the energy of the elementary excitations in the system is obtained by solving the equation

$$\mathrm{tr}\hat{T}_\nu \equiv (\hat{T}_\nu)_{11} + (\hat{T}_\nu)_{22} = 2\cos k_z L_\nu; \quad L_\nu = N_\nu a \tag{11}$$

since the matrix elements depend on $E$.

We wish to calculate properties of the system like the spin wave spectrum, internal energy and specific heat. There arises the problem of the statistics of these excitations [6,7]. The solution of this problem can be found in the following manner. The linearization of the spin-equations of motion is completely equivalent to the transition from spin operators into Bloch's approximation of non-interacting bosons. We shall demonstrate it in the following section.

## 3 System of non-interacting bosons approach

Let us look at the most general Hamiltonian for the superlattice with $N$ layers where each spin has the different value and all interactions are different.

The linearization of spin equation of motion (applied in the construction of transfer matrices) is completely equivalent to the bosonization in terms of Bloch's approximation, i.e. $\hat{S}^+ \simeq \sqrt{2S}\hat{b}$, $\hat{S}^- \simeq \sqrt{2S}\hat{b}^+$, $\hat{S}^z \simeq S - \hat{b}^+\hat{b}$.

The boson Hamiltonian can be written in the form:

$$\hat{H}_B = -\frac{1}{2}\sum_{m}\sum_{n=1}^{N} J_n \sum_{\boldsymbol{\varrho}\,\vec{\delta}_{\parallel}} S_n \Big\{ \Big[\hat{b}^{+}_{m\boldsymbol{\varrho}}(n)\hat{b}_{m\boldsymbol{\varrho}+\boldsymbol{\delta}_{\parallel}}(n) + H.c.\Big] - \hat{b}^{+}_{m\boldsymbol{\varrho}}(n)\hat{b}_{m\boldsymbol{\varrho}}(n) - \hat{b}^{+}_{m\boldsymbol{\varrho}+\boldsymbol{\delta}_{\parallel}}(n)\hat{b}_{m\boldsymbol{\varrho}+\boldsymbol{\delta}_{\parallel}}(n)\Big\} - \sum_{m}\Bigg\{\sum_{n=1}^{N-1} I_{n,n+1}\sum_{\boldsymbol{\varrho}}\Bigg[\sqrt{S_nS_{n+1}}\Big(\hat{b}^{+}_{m\boldsymbol{\varrho}}(n)\hat{b}_{m\boldsymbol{\varrho}}(n+1) + \hat{b}_{m\boldsymbol{\varrho}}(n)\hat{b}^{+}_{m\boldsymbol{\varrho}}(n+1)\Big) - S_n\hat{b}^{+}_{m\boldsymbol{\varrho}}(n+1)\hat{b}_{m\boldsymbol{\varrho}}(n+1) - S_{n+1}\hat{b}^{+}_{m\boldsymbol{\varrho}}(n)\hat{b}_{m\boldsymbol{\varrho}}(n)\Bigg] + \frac{1}{2}I_{N,1}\Bigg[\sqrt{S_1S_N}\Big(\hat{b}^{+}_{m\boldsymbol{\varrho}}(N)\hat{b}_{m+1\boldsymbol{\varrho}}(1) + \hat{b}^{+}_{m-1\boldsymbol{\varrho}}(N)\hat{b}_{m\boldsymbol{\varrho}}(1) + H.c.\Big) - S_N\Big(\hat{b}^{+}_{m+1\boldsymbol{\varrho}}(1)\hat{b}_{m+1\boldsymbol{\varrho}}(1) + \hat{b}^{+}_{m\boldsymbol{\varrho}}(1)\hat{b}_{m\boldsymbol{\varrho}}(1)\Big) - S_1\Big(\hat{b}^{+}_{m\boldsymbol{\varrho}}(N)\hat{b}_{m\boldsymbol{\varrho}}(N) + \hat{b}^{+}_{m-1\boldsymbol{\varrho}}(N)\hat{b}_{m-1\boldsymbol{\varrho}}(N)\Big)\Bigg]\Bigg\}. \quad (12)$$

Let us now perform three-dimensional Fourier-transformation ($N_2 = N_xN_y$, $\boldsymbol{k} = (\boldsymbol{k}_{\parallel}, k_z)$) of boson operators

$$\hat{b}_{m\boldsymbol{\varrho}}(n) = \frac{1}{\sqrt{N_2N_z}}\sum_{\boldsymbol{k}_{||}k_z}\hat{b}_n(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}_{\parallel}\cdot\boldsymbol{\varrho}+i[mL+(n-1)a]k_z}. \quad (13)$$

The quadratic boson Hamiltonian (12) takes the form:

$$\hat{H}_B = \sum_{\boldsymbol{k}}\sum_{n=1}^{N}\Big\{\varepsilon_n\,\hat{b}^{+}_n(\boldsymbol{k})\hat{b}_n(\boldsymbol{k}) + \Big[V_{n,n+1}(\boldsymbol{k})\,\hat{b}^{+}_n(\boldsymbol{k})\hat{b}_{n+1}(\boldsymbol{k}) + H.c.\Big]\Big\} \quad (14)$$

where $\varepsilon_n$ is given by equation (3) from [14], while

$$\varepsilon_1 = 4J_1S_1(1-\gamma_{\boldsymbol{k}_{\parallel}}) + I_{N,1}S_N + I_{12}S_2;$$
$$\varepsilon_N = 4J_NS_N(1-\gamma_{\boldsymbol{k}_{\parallel}}) + I_{N,1}S_{N_1} + I_{N,N-1}S_{N-1}$$

and

$$V_{n,n+1}(\boldsymbol{k}) = -I_{n,n+1}\sqrt{S_nS_{n+1}}\,\mathrm{e}^{ik_za}, \;\; n = 1,2,...,N. \quad (15)$$

This Hamiltonian is diagonalized by the typical unitary transformation

$$\hat{b}_n(\boldsymbol{k}) = \sum_{\alpha=1}^{N} u_{n\alpha}(\boldsymbol{k})\,\hat{a}_{\alpha}(\boldsymbol{k}) \quad (16)$$

leading to the final result

$$\hat{H}_B = \sum_{\boldsymbol{k}}\sum_{\alpha=1}^{N} E_{\alpha}(\boldsymbol{k})\,\hat{a}^{+}_{\alpha}(\boldsymbol{k})\hat{a}_{\alpha}(\boldsymbol{k}) \quad (17)$$

where the magnon excitation energies are determined from the determinant

$$\det\,|\,\overline{V}_{nn'}(\boldsymbol{k}) - E\;\delta_{nn'}\,| = 0 \quad (18)$$

with $\overline{V}_{nn'}(\boldsymbol{k}) = \varepsilon_n\;\delta_{nn'} + V_{nn'}(\boldsymbol{k})$. This is $N\times N$ determinant, which has the following form for $N \geq 3$:

$$\Delta_N(E) = \begin{vmatrix} E-\varepsilon_1 & -V_{12} & 0 & \cdots & 0 & -V^{*}_{1N} \\ -V^{*}_{12} & E-\varepsilon_2 & -V_{23} & \cdots & 0 & 0 \\ 0 & -V^{*}_{23} & E-\varepsilon_3 & -V_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -V^{*}_{N-2,N-1} & E-\varepsilon_{N-1} & -V_{N-1,N} \\ -V_{1N} & 0 & \cdots & 0 & -V^{*}_{N-1,N} & E-\varepsilon_N \end{vmatrix} \quad (19)$$

while for $N = 1$ and $N = 2$ one can use directly the values from the Hamiltonian. One can test, some simple cases like $N = 1,2,3,4$ to see that the energies obtained in two manners i.e. from (11) and $\Delta_N(E) = 0$ are the same. One can also prove that for any layer number $N$, after expanding the determinant, all $k_z$ dependent terms lead to a single term of the type $\cos k_zNa$. Actually we can expand the above determinant leading to the general expression in terms of subdeterminants ($D_{\nu}$ and $\widetilde{D}_{\nu}$):

$$\Delta_N(E) = A_1A_2D_{N-2} - A_1b_3^2D_{N-3} - b_2^2D_{N-2} - b_1^2\widetilde{D}_{N-2} + (-1)^{N-1}\prod_{\sigma=1}^{N} b_{\sigma}\cdot 2\cos Nk_za \quad (20)$$

with the following notations: $A_n = E-\varepsilon_n$, $b_n = I_{n,n-1}\sqrt{S_nS_{n-1}}$, while the subdeterminants $D_{\nu}$ and $\widetilde{D}_{\nu}$ satisfy the following recurrent relations:

$$D_{\nu} = A_{N-\nu+1}D_{\nu-1} - b^2_{N-\nu+2}D_{\nu-2};$$
$$D_0 = 1,\; D_1 = A_N; \quad 2 \leq \nu \leq N-1$$

i.e.

$$\widetilde{D}_{\nu} = A_{N-\nu}\widetilde{D}_{\nu-1} - b^2_{N-\nu+1}\widetilde{D}_{\nu-2};$$
$$\widetilde{D}_0 = 1,\; \widetilde{D}_1 = A_{N-1}, \quad 2 \leq \nu \leq N-1.$$

Using these recurrent relations, one can prove by the mathematical induction that the roots of the equation $\Delta_n(E) = 0$ are the same as the energies obtained from equation (11). This completes our proof of the equivalence of two approaches. This means that one can use the energies obtained from (11) and use the standard Bose-Einstein statistics valid for non-interacting bosons to evaluate all the important quantities.

Using the energies of elementary excitations $E_{\nu}(\boldsymbol{k})$; $(\nu = 1,2,...,N_{\nu})$ defined by equation (11), the contribution to the internal energy at temperature $T$ due to magnons is written as

$$\langle\hat{H}\rangle = U = \sum_{\boldsymbol{k}}\sum_{\nu=1}^{N_{\nu}}\frac{E_{\nu}(\boldsymbol{k})}{\mathrm{e}^{E_{\nu}(\boldsymbol{k})/\theta}-1}, \;\; \theta = k_BT \quad (21)$$

while specific heat in this case (of non-interacting Bosons) is given by

$$C_V = \frac{\partial U}{\partial T} = k_B \frac{\partial U}{\partial \theta} = \frac{k_B}{\theta^2} \sum_{\boldsymbol{k}} \sum_{\nu=1}^{N_\nu} \frac{E_\nu^2(\boldsymbol{k})\, \mathrm{e}^{E_\nu(\boldsymbol{k})/\theta}}{\left(\mathrm{e}^{E_\nu(\boldsymbol{k})/\theta} - 1\right)^2}. \quad (22)$$

The expression for (20) can be written also as

$$\Delta_N(E) = E^N + a_{N-1}^{(N)} E^{N-1} + \ldots + a_1^{(N)} E + a_0^{(N)} \quad (23)$$

where the coefficients $a_j$ depend on both $\boldsymbol{k}_\parallel$ and $k_z$. If one expands $a_0(\boldsymbol{k}_\parallel, k_z)$ up to $k^2$, it can be shown by inspection, that $\boldsymbol{k}$-independent terms cancel exactly, so that the expression for $a_0$ vanishes for $\boldsymbol{k} = 0$. More precisely

$$a_0^{(N)}(\boldsymbol{k}) = a_{0xy}^{(N)}\, (k_x^2 + k_y^2)\, a^2 + a_{0z}^{(N)}\, k_z^2\, c^2. \quad (24)$$

This is in fact the statement of Goldstone theorem, claiming that there exists at least one excitation mode whose energy vanishes for $\boldsymbol{k} = 0$. It is obvious that such mode dictates the low-temperature behavior. Let us denote it by $E_1$:

$$E_1 = \alpha_N\, a^2(k_x^2 + k_y^2) + \beta_N\, c^2 k_z^2, \quad c = Na.$$

Looking at (23), one can see that just last two terms are sufficient to determine $\alpha_N$ and $\beta_N$. Using

$$a_1^{(N)}(0)\left[\alpha_N\,(k_x^2 + k_y^2)\, a^2 + \beta_N\, k_z^2\, c^2\right] + a_{0xy}^{(N)}\,(k_x^2 + k_y^2)\, a^2 - a_{0z}^{(N)}\, k_z^2\, c^2 = 0$$

one obtains

$$\alpha_N = -\frac{a_{0xy}^{(N)}}{a_1^{(N)}(0)}; \quad \beta_N = -\frac{a_{0z}^{(N)}}{a_1^{(N)}(0)}.$$

Using this, the expression for internal energy (21) can be written in the form

$$\frac{U}{N_0} = \frac{1}{N_0} \sum_{\boldsymbol{k}} \sum_{\nu=1}^{N_\nu} \frac{E_\nu(\boldsymbol{k})}{\mathrm{e}^{E_\nu(\boldsymbol{k})/\theta} - 1} \approx \frac{1}{N_0} \sum_{\boldsymbol{k}} \frac{E_1(\boldsymbol{k})}{\mathrm{e}^{E_1(\boldsymbol{k})/\theta} - 1} + \mathcal{O}(\mathrm{e}^{-\Omega_{min}/\theta}) \quad (25)$$

where $\Omega_{min}$ is minimal value of the gap in all other energy branches and $N_0 = N_2 N_z$. Going over from summation to integral over the first Brillouin zone, we obtain

$$\frac{U}{N_0} \approx \frac{1}{N_0} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\alpha_N(q_x^2 + q_y^2) + \beta_N q_z^2}{\mathrm{e}^{(\alpha_N q_\parallel^2 + \beta_N q_z^2)/\theta} - 1} \times dq_x dq_y dq_z + \mathcal{O}(\mathrm{e}^{-\Omega_{min}/\theta}). \quad (26)$$

Integrating over the angles in spherical coordinates and extending the integration limit over radius to infinity (justified for low temperatures), one obtains the result

$$\frac{U}{N_0} = \frac{\xi(5/2)\Gamma(5/2)}{4\pi^2 \alpha_N \sqrt{\beta_N}}\, \theta^{5/2} + \mathcal{O}(\mathrm{e}^{-\Omega_{min}/\theta}) \quad (27)$$

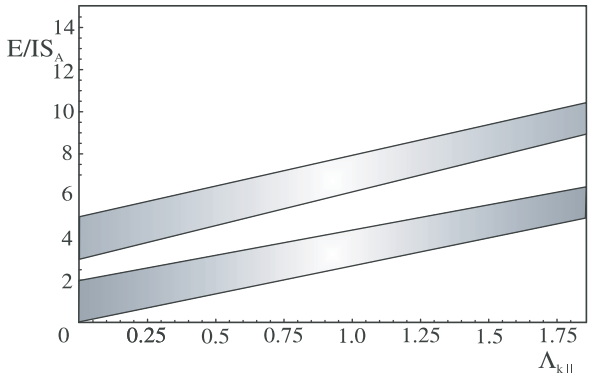


**Fig. 2.** The spin-wave energy $E/IS_A$ in terms of $\Lambda_{\boldsymbol{k}_{||}}$ for $AB$ ($n_A = n_B = 1$). Calculation parameters: $S_A = 1$, $S_B = 3/2$, $J_A = 0.8$, $J_B = 0.4$.

leading to

$$\frac{C_v}{N_0} = \frac{5\xi(5/2)\Gamma(5/2)}{8\pi^2 \alpha_N \sqrt{\beta_N}}\, \theta^{3/2} + \mathcal{O}(\mathrm{e}^{-\Omega_{min}/\theta}). \quad (28)$$

In this way, we have shown analytically that specific heat behaves as $\theta^{\frac{3}{2}}$. One test of the correctness of our result is the fact that for isotropic case ($\alpha_N = \beta_N$) we obtain the standard result for noninteracting bosons [15].

Let us mention that bosonisation within the framework of Holstein-Primakoff approximation was applied to Heisenberg superlattice with various anisotropies [16] but no attempt was made to relate it to the transfer matrix procedure.

Since the energies can be evaluated only numerically, we shall proceed to numerical calculation in the next section.

## 4 Numerical calculations

### 4.1 Spin-wave spectrum

We have decided to plot the renormalized spin-wave energy (divided by the $AB$ interface energy $IS_A$) in terms of $\Lambda_{\boldsymbol{k}_{||}} = 1 - \gamma_{\boldsymbol{k}_{||}}$. We plot here the energy bands of the second and third FG i.e. $AB$ and $ABA$, while choosing $k_y a = 0$. In order to show the advantages of our approach, we performed the calculations for $n_A = n_B = 1$, something that can not be performed with approach proposed in [6–9] demanding at least 3 planes within the cell ($n \geq 3$). The parameters chosen will be $S_A = 1$, $S_B = 3/2$, $J_A = 0.8$, $J_B = 0.4$ while in the third generation we also add $I_A = 0.6$.

Presented figures clearly indicate that for given system parameters, the number of bands exactly corresponds to the index of FG. The limits of the bands in the Figures 2–5 correspond to the value $k_z L_\nu = 0$ and $k_z L_\nu = \pi$, such that we have an alternation from one band to another, following the sequence $0, \pi;\ \pi, 0;\ 0, \pi; \ldots$ starting from the lowest band.

It is very difficult to compare our results with the results of Bezerra et al. [6–9] since they do not quote the

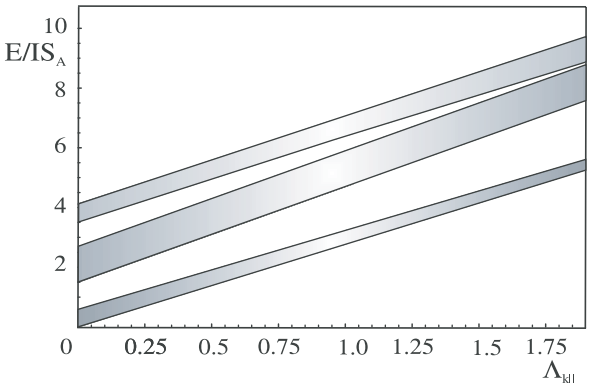


**Fig. 3.** The spin-wave energy $E/IS_A$ in terms of $\Lambda_{\boldsymbol{k}_{||}}$ for $ABA$ ($n_A = n_B = 1$). Calculation parameters: $S_A = 1$, $S_B = 3/2$, $J_A = 0.8$, $J_B = 0.4$, $I_A = 0.6$.

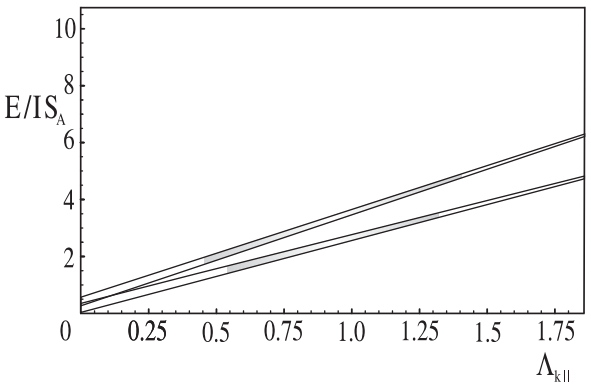


**Fig. 4.** The spin-wave energy $E/IS_A$ in terms of $\Lambda_{\boldsymbol{k}_{||}}$ for $AB$ ($n_A = n_B = 1$) with the biquadratic exchange at the interface. Calculation parameters: $S_A = 1$, $S_B = 3/2$, $J_A = 0.8$, $J_B = 0.4$, $J_{bq}/I = 1.8$.

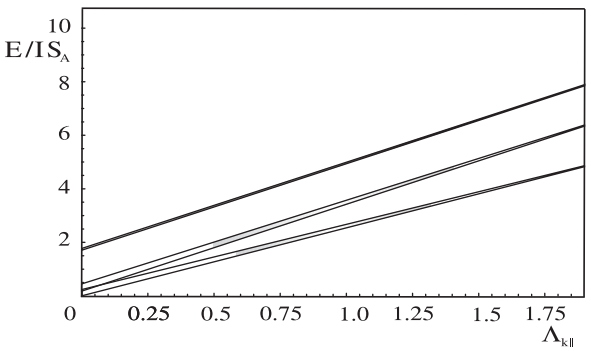


**Fig. 5.** The spin-wave energy $E/IS_A$ in terms of $\Lambda_{\boldsymbol{k}_{||}}$ for $ABA$ ($n_A = n_B = 1$) with the biquadratic exchange at the interface. Calculation parameters: $S_A = 1$, $S_B = 3/2$, $J_A = 0.8$, $J_B = 0.4$, $I_A = 0.6$, $J_{bq}/I = 1.8$.

complete set of parameters. One should stress that the qualitative agreement of our results and the results of mentioned authors, are comparatively good. Yet, it is difficult to follow the comparatively rare manner of plotting the magnon energy for fixed values of wave-vector within the I Brillouin zone as given in the above quoted papers.

### 4.2 Biquadratic coupling

Let us now add the biquadratic coupling ($J_{bq}$) at $AB$ interface following [8]. It is easy to calculate the contribution of biquadratic coupling to bilinear coupling after the linearization. A tedious calculation leads to the additional expression in the equation of motion

$$i\hbar\frac{dS_A^+}{dt} = J_{bq}[S_A(S_B-1)+S_B(S_A-1)](S_A\hat{S}_B^+ - S_B\hat{S}_A^+). \tag{29}$$

This expression just confirms the well-known result [17] that at low temperature this contribution vanishes for $S_A = S_B = 1$. So, although the expressions quoted in reference [7] (Eq. (24)) and reference [18] (Eq. (7)) are incorrect (there appears an additional term "+1"), the general conclusion that the presence of the biquadratic interaction causes the narrowing of the energy bands is confirmed as can be seen from the Figures 4 and 5.

These plots agree with the well-known fact that the ferromagnetic biquadratic coupling has the completely opposite effect to the band-width from of the bilinear ferromagnetic exchange interaction. In fact, biquadratic coupling can yield the changes in the behavior of the magnetic system. It is clear that the increase of the biquadratic interaction can produce the complete compensation the effect of bilinear exchange interaction, causing the vanishing of the interlayer coupling between the FG (in interfaces). This results in vanishing of the FG-s, which constitute the Fibonacci superlattices, so they behave as independent blocks. The further increase of the biquadratic interaction leads to the spin reorientation in different layers [19,20].

### 4.3 Magnon specific heat

There are several types of excitations in solids whose contribution can be distinguished during the measurement of the specific heat. In particular, the contribution to the specific heat can come from phonons, electrons, magnons, excitons, etc. In this subsection we shall focus our attention only to the magnon contribution to the constant volume specific heat. The specific heat is calculated from the expression (22), which is exact for the non-interacting bosons. In this way, we calculated the magnon specific heat for the second ($AB$), third ($ABA$), fifth ($ABAABABA$) and eight generation ($ABAABABAABAAB$) for the same set of parameters, as above, up to $\theta/IS_A \cong 1$, which agrees with mean-field estimate of $\theta_C/3$, where boson approach is valid.

One can see that the specific heat increases with the order of generation. Also, typically for the system under study, the specific heat vanishes for $\theta \to 0$, contrary to the results [6,7] which contain some peculiar oscillatory behavior of the specific heat and its finite value for vanishing temperature.

Since we have proved that the linearization of spin equations is equivalent to the lowest order Bloch's approximation, one should expect the behavior corresponding to

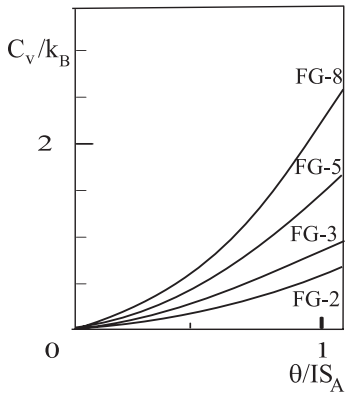


**Fig. 6.** Specific heat (per unit cell) in terms of renormalized temperature for second, third, fifth and eight FG, in the temperature range where boson approximation is justified.

the non-interacting bosons. However, an additional verification of our approach comes from the fact that the curves at Figure 6 agree within the line width with the curves calculated from equation (28) for the same set of parameters (for $N = 3$ and $N = 4$, where one can obtain analytic expressions of $\alpha_N$ and $\beta_N$).

There were previous attempts to obtain to this specific heat numerically ([21], although the quality of the fit for low temperatures can be questioned.

## 5 Conclusion

In this paper we investigated and analysed the layered ferromagnetic quasi-periodic Fibonacci superlattices using the transfer matrix method in the low-temperature limit. Of course, introduction of free boundary conditions would be more realistic, yet it would introduce also surface excitations, which would be the subject of an independent study. We studied the case of ferromagnetic bilinear and biquadratic coupling in interface and demonstrated by construction that the transfer matrix of the given $\nu$th FG can not be obtained as the product of the previous two ones, $\nu - 1$ and $\nu - 2$th FG. This result is the consequence of the existence of different boundary matrices and it is completely different from the results obtained by the authors of the series of papers published on the subject [1,6–9]. The spectrum of the elementary magnon excitations was determined by the numerical analysis.

Also, we have shown, that the biquadratic ferromagnetic coupling has the opposite trend compared to bilinear exchange interaction. The increase of the biquadratic interface interaction can cause the compensation of the bilinear coupling leading finally to independence of the blocks building Fibonacci superlattices so that in the case of a few ferromagnetic layers one can observe the quasi two-dimensional behavior.

The same results were obtained also after the bosonization in Bloch's approximation. The equivalence of the transfer-matrix method and boson procedure was explicitly demonstrated. The low-temperature internal energy of the magnon system and its contribution to the specific heat was evaluated both analytically and numerically. We have demonstrated that close to absolute zero, the specific heat vanishes as $\sim T^{3/2}$ characteristic for 3D isotropic ferromagnetic systems as a consequence of existing Goldstone-mode.

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